Thin Airfoil in Supersonic Flow

Consider an airfoil of chord length c placed in a uniform supersonic[1] stream, U_∞ at a small angle of attack α as shown in the Figure 1.[2] Let

$$y = f_u(x) \quad 0 \le x \le c, \tag{1}$$
$$y = f_\ell(x) \quad 0 \le x \le c, \tag{2}$$

represent the upper and lower surfaces of the airfoil. In linearized thin airfoil theory, the velocity is potential and can be represented by

$$\mathbf{V} = U_\infty \mathbf{e}_x + \nabla\phi, \tag{3}$$

where U_∞ is the upstream uniform velocity and ϕ is the perturbation potential due to the presence of the airfoil. $\nabla\phi = \{\partial\phi/\partial x, \partial\phi/\partial y\} = \{u, v\}$, the x and y components of the perturbation velocity. The velocity potential ϕ satisfies the following equation

$$\beta^2 \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} = 0, \tag{4}$$

where $\beta^2 = M^2 - 1$. Along the airfoil surface we assume impermeability which, within the approximation of the thin airfoil theory, gives

$$(v)_{u,\ell} = U_\infty\left(\frac{df_{u,\ell}}{dx}\right). \tag{5}$$

At large distance, the velocity field is assumed to be finite.

1. Show that (4) is hyperbolic and write the equations for the characteristics $C^{\pm}$. Integrate theses equations and draw the characteristics straight lines with slopes $\pm 1/\beta$. In aerodynamics, the supersonic characteristic lines are known as the Mach lines and they make the angle $\pm sin^{-1} 1/M$ with x axis.

2. Write the equations to be satisfied by $\{u, v\}$ and integrate these equations. Write the general solution for (4).

3. Let D_+ and D_- be the upper and lower domains defined by the Mach lines passing by the leading and trailing edges shown in Figure 1. Show that since the flow is supersonic no perturbations can travel upstream and therefore,

$$\phi_u = \phi_u(x - \beta y) \ \text{ in } D_+, \tag{6}$$
$$\phi_\ell = \phi_\ell(x + \beta y) \ \text{ in } D_-. \tag{7}$$

[1] A flow is supersonic if its velocity is higher than the speed of sound, i.e., the Mach number $M > 1$.

[2] Since $\alpha << 1$, $sin\alpha \approx \alpha$ and $cos\alpha \approx 1$.

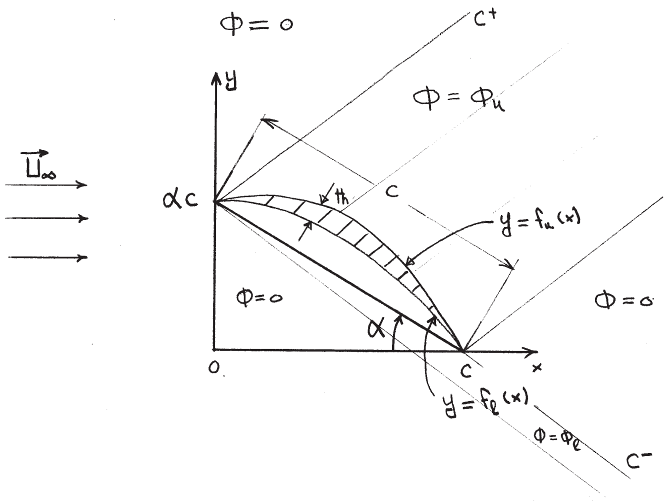

Figure 1: Airfoil in supersonic flow.

and $\phi \equiv 0$ outside D_+ and D_-.

4. Noting that

$$c_p = -\frac{2}{U_\infty}\frac{\partial \phi}{\partial x}, \tag{8}$$

show that the lift coefficient is given by

$$C_L = \frac{4\alpha}{\sqrt{M^2 - 1}}. \tag{9}$$

Compare with the subsonic lift formula:

$$C_L = \frac{2\pi}{\sqrt{1 - M^2}}(1 + 0.77\theta)(\alpha + 2m), \tag{10}$$

where θ is the thickness ratio and m, the camber.

SOLUTION

1. Equation (3) is hyperbolic. The equation for the characteristics is

$$\beta^2 (\frac{dy}{dx})^2 - 1 = 0. \tag{11}$$

This gives

$$I^+: \quad \frac{dy}{dx} = \frac{1}{\beta}, \tag{12}$$

$$I^-: \quad \frac{dy}{dx} = -\frac{1}{\beta}. \tag{13}$$

or in integrated from

$$I^+ : x - \beta y = \lambda, \tag{14}$$

$$I^- : x + \beta y = \mu. \tag{15}$$

If the Mach wedge half angle is δ, since $tan\delta = 1/\beta$, $\delta = sin^{-1} 1/M$.

2. The equations for $u = \phi_x$ and $v = \phi_y$ are

$$II^+: \quad du - \frac{dv}{\beta} = 0, \tag{16}$$

$$II^-: \quad du + \frac{dv}{\beta} = 0. \tag{17}$$

or, in integrated from,

$$II^+ : \phi_x - \frac{\phi_y}{\beta} = 2F'(\lambda), \tag{18}$$

$$II^- : \phi_x + \frac{\phi_y}{\beta} = 2G'(\mu). \tag{19}$$

Hence the general solution for (3) is of the form

$$\phi = F(x - \beta y) + G(x + \beta y). \tag{20}$$

3. In supersonic flows, information cannot be transmitted outside the Mach wedge. Therefore, for the region bounded by the suction surface of the airfoil and the two characteristics passing by the leading edge ($x - \beta y = -\beta\alpha c$)and trailing edge ($x - \beta y = c$), the function ϕ depends only on the parameter $\lambda = x - \beta y$. We denote this function by ϕ_u. Similarly, for the region bounded by the pressure surface of the airfoil and the two characteristics passing by the leading edge ($x + \beta y = \beta\alpha c$) and trailing edge ($x + \beta y = c$), the function ϕ depends only on the parameter $\mu = x + \beta y$. We denote this function by ϕ_ℓ. Therefore,

$$\phi_u = \phi_u(x - \beta y) \text{ in } D_+, \tag{21}$$

$$\phi_\ell = \phi_\ell(x + \beta y) \text{ in } D_-. \tag{22}$$

and $\phi \equiv 0$ outside D_+ and D_-.

4. The impermeability condition along the airfoil pressure and suction surfaces is

$$(\frac{v}{U_\infty})_{u,\ell} = \frac{df_{u,\ell}}{dx}. \tag{23}$$

This leads to

$$\begin{aligned} \frac{\partial \phi_u(x-\beta y)}{\partial y} &= U_\infty \frac{df_u}{dx}, & (24) \\ \frac{\partial \phi_u(x+\beta y)}{\partial y} &= U_\infty \frac{df_\ell}{dx}. & (25) \end{aligned}$$

However,

$$\begin{aligned} \frac{\partial \phi_u}{\partial y} &= -\beta \frac{d\phi_u}{d\lambda} = -\beta \frac{\partial \phi_u}{\partial x}, & (26) \\ \frac{\partial \phi_\ell}{\partial y} &= +\beta \frac{d\phi_\ell}{d\mu} = \ \beta \frac{\partial \phi_\ell}{\partial x}. & (27) \end{aligned}$$

Substituting (26) and (27) into (24) and (25), respectively and integrating, we get

$$\begin{aligned} \phi_u &= -\frac{U_\infty}{\beta} f_u(x-\beta y), \ \text{ in } D_+, & (28) \\ \phi_\ell &= +\frac{U_\infty}{\beta} f_\ell(x+\beta y), \ \text{ in } D_-. & (29) \end{aligned}$$

$$C_L = \frac{1}{c}\int_0^c (-c_{p_u} + c_{p_\ell})dx. \tag{30}$$

Substituting (8) into(30), we get

$$C_L = -\frac{2}{U_\infty c}\int_0^c (-\frac{\partial \phi_u}{\partial x} + \frac{\partial \phi_\ell}{\partial x})dx. \tag{31}$$

Using the expressions for ϕ_u and ϕ_ℓ given in (28) and (29) and integrating, we get

$$C_L = -\frac{2}{c\beta}[f_u + f_\ell]_0^c = \frac{4\alpha}{\sqrt{M^2-1}}. \tag{32}$$

Comparing (32 and 10) shows that within the approximation of thin airfoil theory, the supersonic lift coefficient does not depend on the thickness and camber of the airfoil. We also note that the supersonic lift coefficient has a small slope (4) with the angle of attack than the subsonic lift coefficient (2π).